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# Weighted Geodetic Convex Sets in A Graph

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***Abstract:*** *Let tt : (V, E, W ) be a finite, connected, weighted graph without loops and multiple edges. In a weighted graph each arc is assigned a weight by the weight function W : E → R+. A u v path P in tt is called a weighted u v geodesic if the weighted distance between u and v is calculated along P. The strength of a path is the minimum weight of its arcs, and length of a path is the number of edges in the path. In this paper, we introduce the concept of weighted geodesic convexity in weighted graphs. A subset W of V (tt) is called weighted geodetic convex if the weighted geodetic closure of W is W itself.The concept of weighted geodetic blocks are introduced and discussed some of their properties. The notion of weighted geodetic boundary and interior points are included.*
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## I. Introduction

Weighted graph theory has numerous applications in various fields like cluster- ing analysis, operations research, database theory, network analysis, information theory, etc. During the last few decades, the fast development of a number of discrete and combinatorial mathematical structures has lead to the study of generalizations of a number of classical concepts from continuous mathematics. Among them, the concept of convex set of a metric space plays a key role. The reason is, all connected graphs can be seen as metric spaces just by considering their shortest paths. This fact has lead to the study of the behavior of these structures as convexity spaces. An ordinary graph is a weighted graph with unit weight assigned for all arcs. That means an ordinary graph is particular weighted graph. All the practical problematic situations which can be solved by using any graph theory technique can only be medelled by a weighted graph. This fact is the main motivation for this work. In this paper, we define the weighted geodesic convexity in weighted graphs.

Let $G : (V,E)$ be a crisp ordinary graph. Then the distance between two nodes $u$ and $v$ of $G$ is defined as the length of a $u - v$ geodesic [1, 2]. A shortest $u - v$ path is called a $u - v$ geodesic [1, 2] The length of a path is the number of edges

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present in the path [1, 2]. Strength of path $P = v_0 e_1 v_1 e_1 v_2 ... e_n v_n$, is denoted and defined by $S(P) = w(e_1) \wedge w(e_2) \wedge ... \wedge w(e_n)$[10]. The length of the path $P$ is the number of arcs present in $P$ . The weighted distance between two nodes $u$ and $v$ in $G$ is defined and denoted by $d_W(u, v) = \wedge_P \{l(P) * S(P)/P$ is a $u - v$ path, $l(P)$ is the strength and $S(P)$ is the strength of $P\}$ [11]. A $u - v$ path $P$ is called a weighted $u - v$ geodesic if $d_W(u, v) = l(P) * S(P)$ [12]. Several authors have made remarkable contributions to weighted graph theory. They include Paul Er- dos, Bondy and Fan [3, 4], Broersma, Zhang and Li [6], Mathew and Sunitha, Sampathkumar[7], Soltan[9].

## II. Weighted geodetic convexset

In this section, we define the concept of weighted geodetic convex sets and discuss some of their properties.

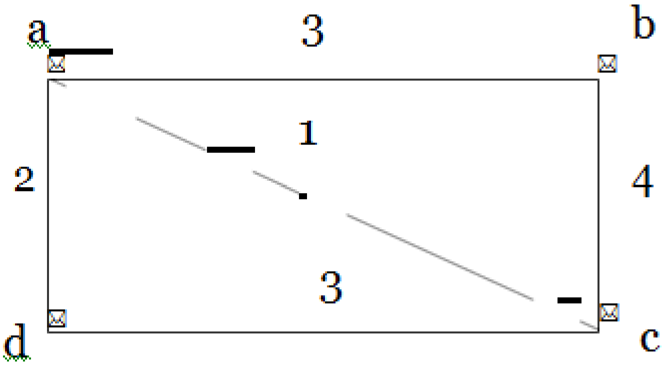
**Definition 2.1.** *Let $G : (V,E,W)$ be a connected weighted graph without loops and multiple edges. Let $u$, $v$ be any two nodes of $G$. A $u-v$ path $P$ is called a weighted $u - v$ geodesic if $d_W(u, v) = S(P) * l(P)$. This means a $u - v$ path $P$ is called a weighted $u - v$ geodesic if the weighted distance between $u$ and $v$ is calculated along the path $P$.* □

**Definition 2.2.** *For any two nodes u and v of G, the weighted geodetic closed interval $I_W[u,v]$ is the set of all nodes in all weighted $u - v$ geodesics including u and v.* □

**Definition 2.3.** *Let $G : (V,E,W)$ be a connected weighted graph without loops and multiple edges and let $S \subseteq V$. The union of all geodetic closed intervals $I_W[u,v]$ over all pairs $u, v \in S$ is called the weighted geodetic closure of $S$. It is* *denoted by $I_W (S)$.*

**Definition 2.4.** *Let $G : (V,E,W)$ be a connected weighted graph without loops and multiple edges. Any subset $S$ of $V(G)$ is called weighted geodetic convex if $I_W (S) = S$.*

**Example 2.1.**




(**Figure.1:** A weighted graph)

The weighted distance matrix for the above graph (Figure. 1) is given below.

$$\begin{pmatrix} 0 & 2 & 1 & 2 \\ 2 & 0 & 2 & 3 \\ 1 & 2 & 0 & 2 \\ 2 & 3 & 2 & 0 \end{pmatrix}$$

In figure.1, the path $a - c - b$ is a weighted $a - b$ geodesic. Note that it is unique. The direct edge $(a, c)$ is a weighted $a - c$ geodesic. It is also unique. Again $I_W[a,d] = \{a,c,d\}$, $I_W[a,c] = \{a,c\}$, $I_W[a,b] = \{a,c,b\}$. If $S = \{a, b\}$,
$I_W[S] = \{a,b,c\} \neq S$. Thus $S$ is not weighted geodetic convex. But if $S = \{a,c\}$,
then $I_W [S] = S$, which proves $S$ is weighted geodetic convex.

The following preposition is obvious. The proof is omitted.

**Preposition 2.1.** *Let $G(V, E, W)$ be a connected weighted graph. Then the empty set $\Phi$, the full node set $V(G)$ and all singletons $V(G)$ are weighted geode- tic convex.*

By a nontrivial weighted geodetic convex set, we mean a weighted geodetic con- vex set $S$ with $2 \leq |S| < |V(G)|$.
In the next theorem, we show that the arbitrary intersection of weighted geodetic convex sets is again weighted geodetic convex.

**Theorem 2.1.** *The intersection of two weighted geodetic convex sets is again weighted geodetic convex.*

**Proof.** Let $G(V,E,W)$ be a connected weighted graph. Let $S$ and $T$ be any two weighted geodetic convex sets of $V(G)$. We have to prove that $S\ T$ is weighted geodetic convex. Let $u$ and $v$ be any two nodes of $S \cap T$. This means $u$ and $v$ are
two nodes in both $S$ and $T$, which are weighted geodetic convex. Therefore all nodes in all weighted $u - v$ geodesics are both in $S$ and $T$, and hence in $S \cap T$. This shows that $S \cap T$ is weighted geodetic convex.

From figure. 1, it is clear that union of two weighted geodetic convex sets is not weighted geodetic convex. Consider $S = a$ and $T = b$. $S\ T = a, b$, which is not weighted geodetic convex.

## III. Weighted geodetic blocks and their character- ization

In this section, we introduce the concept of weighted geodetic blocks. Some nec- essary conditions and a characterization is also included. By a complete weighted
graph $G$, we mean that the underlying graph $G^*$ of $G$ is complete.

**Definition 3.1.** *Let $G:(V,E,W)$ be a connected weighted graph and let $H$ be a complete weighted subgraph of $G$. Then $H$ is called a weighted geodetic block of*
*$G$ if there exists no edge $e = (u,v)$ in $H$ such that $w(e) \geq d_W(u,v)$.*
**Remark 3.1.** From the definition of weighted geodetic blocks, it is clear that all edges $e=(u,v)$ in a weighted geodetic block are the only weighted $u-v$ geodesics.
**Example 3.1.**

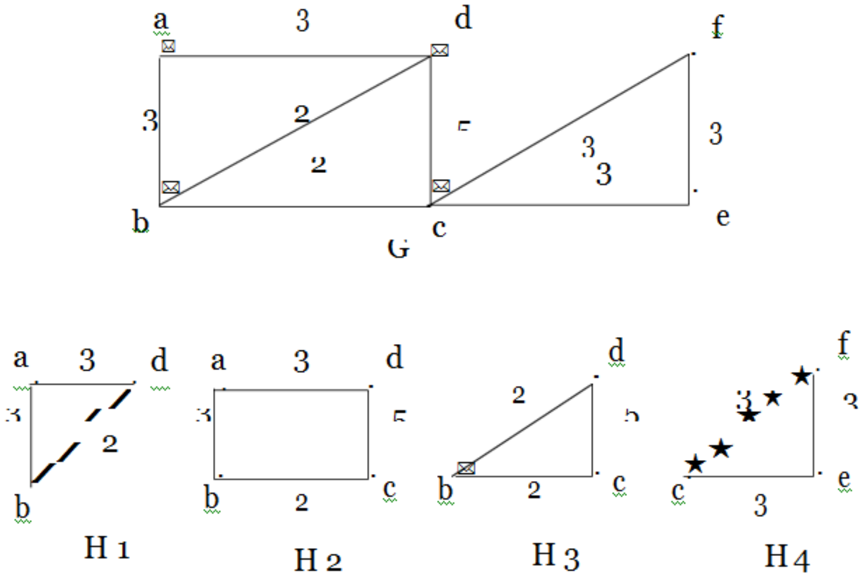



**Figure.2:** A weighted graph and its geodetic blocks

In figure. 2, $H1$ and $H4$ are weighted geodetic blocks of $G$, but $H2$ and $H3$ are not. Even though all the arcs in $H2$ are weighted geodesics between their end

nodes, its underlying graph $H2^*$ is not complete. In $H3$, the edge $(c, d)$ is not a weighted $c - d$ geodesic.
The following theorem is a necessary condition for a weighted geodetic block.
**Theorem 3.1.** *Let $G:(V,E,W)$ be a connected weighted graph. Let $H$ be a complete weighted subgraph of $G$. If $H$ is a weighted geodetic block of $G$, then $V(H)$ is a weighted geodetic convex set of $G$.*
**Proof.** Let $G:(V,E,W)$ be a connected weighted graph and $H$ be a complete weighted subgraph of $G$. Suppose that $H$ is a weighted geodetic block of $G$. We have to prove that $V(H)$ is a weighted geodetic convex set of $V$. Since $H$ is a weighted geodetic block of $G$, there exists no edge $e = (u, v)$ in $H$ such that $w(e)$ $d_W(u,v)$. This means every edge $e = (u,v)$ in $H$ is the only weighted $u$ $v$ geodesic in $H$. Therefore the nodes in all weighted $u$ $v$ geodesics are exactly $u$ and $v$. This is true for any pair of nodes in $V(H)$. Hence $V(H)$ is a weighted geodetic convex set of $V$.
This theorem is not sufficient. That is , if $H$ is a complete weighted subgraph of $G$ such that $V(H)$ is a weighted geodetic convex subset of $V$, then $H$ need not be a weighted geodetic block of $G$. This is explained in the following example.

**Example 3.2.**

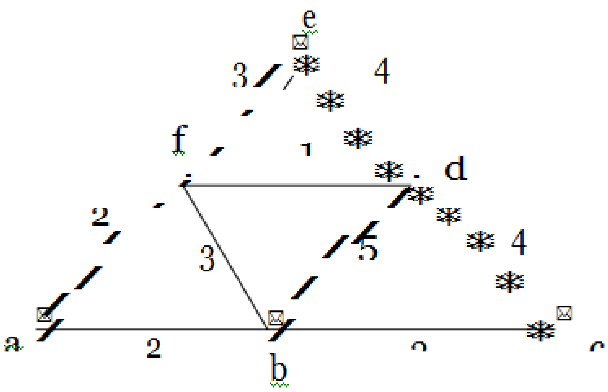



**Figure.3:** A weighted graph

In figure. 3, the vertex induced sub graph induced by the nodes $d, e$ and $f$ is not a weighted geodetic block, even though $\{d, e, f\}$ is a weighted geodetic convex set of $V$. Note that the edges $(e, f)$ and $(d, f)$ are not weighted geodesics between their end points.

The sequential deletion of nodes from a weighted geodetic block results in a nested sequence of weighted geodetic blocks. This fact is explained in the following the- orem.

**Theorem 3.2.** *Every node deleted subgraph of a weighted geodetic block is again a weighted geodetic block.*

**Proof.** Let $G : (V, E, W)$ be a connected weighted graph and $H$ be a weighted geodetic block of $G$. Let $u$ be any node of $H$. Consider the node deleted subgraph $H - u$. We have to prove that $H - u$ is again a weighted geodetic block of $G$. Let $e = (x, y)$ be any edge of $H - u$. Clearly $x \neq u \neq y$, and since $(x, y)$ is an edge of $H$, which is weighted geodetic convex, $e$ is the only weighted $x - y$ geodesic in $H - u$. This is true for any edge in $H - u$. Thus $H - u$ is a weighted geodetic block of $G$.

By the above theorem, it is clear that, if we have a weighted geodetic block with order $p$, then there exists at least one weighted geodetic blocks of orders 1, 2, 3,..., $(p - 1)$ each.

In the next theorem, we characterize weighted geodetic blocks of a connected weighted graph $G$.

**Theorem 3.3.** *Let $G : (V, E, W)$ be a connected weighted graph and let $H$ be a complete weighted subgraph of $G$. Then $H$ is a weighted geodetic block of $G$ if and only if {u,v} is weighted geodetic convex for every two nodes u, v in H.*

**Proof.** Let $G : (V, E, W)$ be a connected weighted graph and let $H$ be a complete weighted subgraph of $G$.

Suppose that $H$ is a weighted geodetic block of $G$. Let $u$, $v$ be any two nodes of $H$. We have to prove that $\{u, v\}$ is weighted geodetic convex. Since $H$ is a weighted geodetic block, every pair of nodes are adjacent and end every edge is the only weighted geodesic between its end nodes. In particular the nodes $u$ and $v$ are adjacent and $(u, v)$, is the only weighted $u - v$ geodesic. This proves that $\{u,v\}$ weighted geodetic convex.

Conversely suppose that for any two nodes $u$, $v$ in $H$, $\{u, v\}$ is weighted geodetic convex. We have to prove that $H$ is a weighted geodetic block of $G$. That is to prove every edge in $H$ is the only weighted geodesic between its end nodes. By the assumption, each arc $e = (u,v)$ is a weighted geodesic between its end nodes. Now it remains to prove the uniqueness of the geodesic. Suppose the contrary. Let there be another weighted $u - v$ geodesic in $H$, say, $P$. Suppose that $P$ has $l$ edges. Let the minimum weight in $P$ be $\theta$. Let $w(e) = k$. Since $P$ and $e$ are two weighted $u - v$ geodesics, $S(P) \cdot l(P) = S(e) \cdot l(e)$. This means $l\theta = k$. Since $H$ is complete, we can construct another weighted $u - v$ geodesic with $m$ edges, where $m < l$ and strength $\theta$. Thus we get $l\theta = k = m\theta$, which implies $l = m$, a contradiction. Hence our assumption is wrong. So all the edges in $H$ are the only weighted geodesics between their end nodes. Thus $H$ is a weighted geodetic block.

**Corollary 3.1.** A complete connected weighted graph $G^* : (V, E, W^*)$ is a weighted geodetic block if and only if it has $(2^{|V|} - (|V| + 2))$ nontrivial weighted geodetic convex sets.

**Proof.** Let $G : (V, E, W)$ is a complete connected weighted graph. Suppose that $G$ is a weighted geodetic block. Then by the above theorem (Theorem 3.3), for any two nodes $u$ and $v$, $\{u, v\}$ is weighted geodetic convex. If we extend this set by adding any number of nodes, the resultant set is again a weighted geodetic convex set. Thus there are $2^{|V|}$ weighted geodetic convex. Therefore the number of nontrivial weighted geodetic convex sets is $(2^{|V|} - (|V| + 2))$.

Conversely if there are $(2^{|V|} - (|V| + 2))$ number of nontrivial weighted geodetic convex sets of $G$, all the two elemented subsets of $V(G)$ are weighted geodetic convex. By the above theorem (theorem 3.3.), $G$ is weighted geodetic convex.

in the following theorem, we can see that the vertex set of any weighted tree can be considered as the nested union of subsets, all of which are weighted geodetic convex.

**Theorem 3.4.** *Let $G : (V, E, W)$ be a weighted tree, then there exists a sequence of sets $V = V_n \supset V_{n-1} \supset, ..., \supset V_1$, where for each $i$, $V_i$ is weighted geodetic convex and $|V_i| = n_i$*

**Proof.** Let $G : (V, E, W)$ be a weighted tree with $n$ nodes. Then between any two nodes of $G$, there exists only one path, and hence it is the unique weighted geodesic between them. So all nodes in all weighted geodesics between any two nodes are again in $V(G)$. Therefore $V(G) = V_n$ is weighted geodetic convex and $|V_n| = n$. Let $v_1$ be a pendent node of $G$ and set $V_{n-1} = V_n - \{v_1\}$. Then clearly the vertex induced subgraph of $G$ by the set $V_{n-1}$ is again a weighted tree. By the above same argument, we see that $V_{n-1}$ is weighted geodetic convex and

$|V_{n-1}| = (n-1)$. Let $v_2$ be a pendent node of $G - v_1$, set $V_{n-2} = V_{n-1} - v_2$.
We can easily prove $V_{n-2}$ is weighted geodetic convex, and $|V_{n-2}| = (n-2)$. Continue the above procedure of deleting pendent nodes until we get a singleton set. So finally we get the nested sequence $V = V_n \supset V_{n-1} \supset, \ldots, \supset V_1$.

**Remark 3.1.** The converse of the above theorem is not true. That means even though we can find a nested sequence of node sets satisfying the property in the condition, the graph may not be a weighted tree. It can be a weighted geodetic block also.

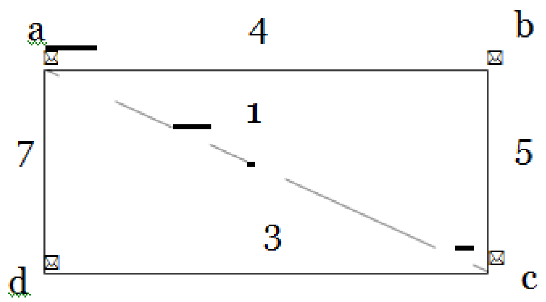
## IV. Weighted geodetic boundary and interior nodes

In this section, we define the boundary and interior nodes of a weighted geodetic convex set. Some of their properties are discussed.

**Definition 4.1.** *Let $G : (V,E,W)$ be a connected weighted graph and $S$ be a weighted geodetic convex set of $G$. A node $u \in S$ is called a weighted geodetic boundary node of $S$ if and only if $S - \{u\}$ is weighted geodetic convex*

**Definition 4.2.** *Let $G : (V,E,W)$ be a connected weighted graph and $S$ be a weighted geodetic convex set of $G$. A node $u \in S$ is called a weighted geodetic interior node of $S$ if and only if $S - \{u\}$ is not weighted geodetic convex*

**Example 4.1.**




(**Figure.4:** Boundary node and interior node)

In the above example (Figure.4), $S = \{a, b, c\}$ is a weighted geodetic convex set. Node $c$ is a weighted geodetic interior node of $S$, since $S - \{c\}$ is not weighted geodetic convex. But node $b$ is a weighted geodetic boundary node of $S$, as $S - \{b\}$ is weighted geodetic convex.

In the following two theorems, we characterize the boundary and interior nodes of a weighted geodetic convex set.

**Theorem 4.1.** *Let $G : (V,E,W)$ be a connected weighted graph, and $S$ be a weighted geodetic convex set of $G$. Then a node $u \in S$ is a weighted geodetic boundary node of $S$ if and only if $u$ does not lie on any weighted $v-w$ geodesic for all $v,w \in S-\{u\}$*

**Proof.** Let $G : (V,E,W)$ be a connected weighted graph, and $S$ be a weighted geodetic convex set of $G$. Let $u \in S$. Suppose that $u$ is a weighted geodetic boundary node of $S$. We have to prove that $u$ does not lie on any weighted $v-w$ geodesic for all $v, w \in S-\{u\}$. Since $u$ is a weighted geodetic boundary node of $S$, $S-\{u\}$ is weighted geodetic convex. Let $v,w \in S-\{u\}$. Since $S-\{u\}$ is weighted geodetic convex, all the nodes in all weighted $v-w$ geodesics are in $S-\{u\}$ itself. This means all the weighted $v-w$ geodesics are independent of $u$. Hence $u$ does not lie in any weighted $v-w$ geodesic for all $v,w \in S-\{u\}$.

Conversely suppose that $u \in S$ and $u$ does not lie in any weighted $v-w$ geodesic for all $v,w \in S-\{u\}$. We have to prove that $u$ is a weighted geodetic boundary node of $S$. It is enough, if we prove that $S-\{u\}$ weighted geodetic convex. Let $v,w \in S-\{u\}$. By the assumption, all the weighted $v-w$ geodesics are free from $u$. Also $(S - \{u\}) \cup \{u\} = S$, which is weighted geodetic convex. Hence all the nodes in all weighted geodesics between any two nodes in $S - \{u\}$ lie in $S - u$ itself, which proves $S - u$ is weighted geodetic convex. So $u$ is a weighted geodetic boundary node of $S$.

**Theorem 4.2.** *Let $G : (V,E,W)$ be a connected weighted graph, and $S$ be a weighted geodetic convex set of $G$. Then a node $u \in S$ is a weighted geodetic interior node of $S$ if and only if $u$ lies in at least one weighted $v-w$ geodesic for any $v,w \in S$*

$-\{u\}$

**Proof.** Let $G:(V,E,W)$ be a connected weighted graph, and $S$ be a weighted geodetic convex set of $G$. Let $u \in S$. Suppose that $u$ is a weighted geodetic interior node of $S$. We have to prove that $u$ lies in at lest one weighted $v-w$ geodesic for any $v, w \in S-\{u\}$. Since $u$ is a weighted geodetic interior node of $S$, $S-\{u\}$ is not weighted geodetic convex. Let $v, w \in S-\{u\}$. Since $S-\{u\}$ is not weighted geodetic convex, at least one node in any one of the weighted $v-w$ geodesic does not belong to $S-\{u\}$. At the same time $(S-\{u\})\cup\{u\}=S$, which is weighed geodetic convex. This means at least one of the weighted $v\,w$ geodesic contains the node $u$.

Conversely suppose that $u\in S$ and $u$ lies in any weighted $v\,w$ geodesic for some $v, w\in S-\{u\}$. We have to prove that $u$ is a weighted geodetic interior node of $S$. It is enough, if we prove that $S-\{u\}$ is not weighted geodetic convex. Let $v, w\in S-\{u\}$. By the assumption, there exists at least one weighted $v\,w$ geodesic which contains $u$. Also $(S-\{u\})\cup\{u\}=S$, which is weighted geodetic convex. Hence $S-\{u\}$ is not weighted geodetic convex. So $u$ is a weighted geodetic interior node of $S$.

In the following theorem, we characterize the boundary and interior nodes of a weighted geodetic block.

**Theorem 4.3.** *Let $G:(V,E,W)$ be a connected weighted graph, and let $H$ be a weighted geodetic block of $G$. Then every node of $H$ is a weighted geodetic bound- ary node of $V(H)$.*

**Proof.** Let $G:(V,E,W)$ be a connected weighted graph and let $H$ be a weighted geodetic block of $G$. Let $u$ be any node of $H$. We have to prove that $u$ is a weighted geodetic boundary node of $V(H)$. It is enough, if we prove $V(H)-\{u\}$ is weighted geodetic convex. Since a weighted geodetic block is a complete weighted graph structure and each of its arcs are unique weighted geodesics between their end nodes, by theorems 3.2 and 3.1, $V(H)-\{u\}$ is weighted geodetic convex. This proves that $u$ is a weighted geodetic boundary node of $V(H)$. Since $u$ is arbitrary, the proof is completed.

**Corollary 4.1.** *Let $G:(V,E,W)$ be a connected weighted graph and let $H$ be a weighted geodetic block of $G$. Then no node of $H$ is a weighted geodetic interior node of $V(H)$.*

## V. Conclusion

In this article, the authors made an attempt to generalize the concept of con- vexity. This generalization is done by using the help of weighted distance in weighted graphs. The concepts of weighted geodesics, weighted geodetic convex- ity, weighted geodetic blocks, boundary and interior nodes of a weighted geodetic convex sets are introduced. Characterization for weighted geodetic blocks and boundary and interior nodes are also presented. We have proved that, for a weighted geodetic block, all nodes are boundary nodes and no node is an interior node.

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